The New Integral Transform ''Abaoub-Shkheam transform''

Ali E. Abaoub, Abejela S. Shkheam
Department of mathematics - Sabratha University, faculty of science, Libya.
abaoub59@outlook.com
eshehkam999@gmail.com

Abstract-In this paper a new integral transform namely Abaoub-Shkheam transform was introduced. Fundamental properties of this transform were derived and presented such as linearity, change of scale, and first translation or shifting. It is proven and tested to covering equation for temperature distributions in a semi-infinite bar. This transform may solve some different kind of integral and differential equations and it competes with other known transforms like Sumudu and Yang Transform.

Keywords: Integral transform – Laplace transform – Linear differential equations.

I. Introduction

One of the most effective tools for solving problems in physics and engineering is using the transform method to obtain a solution for a given differential equations or integral equation by means of inverse transformation. Among these Transforms are the Fourier [1], Laplace [2], Hankel [3], Mellin [3] Sumudu [4], and Elzaki [5-6]. The importance of an integral transforms is that they provide powerful operational methods for solving initial value problems and initial- boundary value problem for linear differential and integral equations.

Recently, in 2013, Khalid S. Aboodh introduces “Aboodh Transform” and applies for solving ordinary differential equations [7-8]. It is also used to solve partial differential equations [9], Heat equations [10]. Also in 2019, Shuhu was introducing a new transform which is a generalization of the Laplace and the Sumudu integral transforms for solving differential equations in the time domain [11].

In view of many interesting properties which make visualization easier, we introduced a new integral transform, termed as "Abaoub – Shkheam" Transformation, briefly Q – Transform and applied it to the solution of temperature distributions problem (heat partial differential equations). Subsequently, we derived the Q-Transform of different functions and derivatives used in engineering problems. The plan of the paper is as follows: In section 2, we introduce the basic idea of Q- transform, some fundamental properties in 3, then Application to solve partial differential equation in 4 and conclusion in 5, respectively.

1. Abaoub- Shkheam Transform "Q – Transform":

Definition:

Let $f(t)$ be a function defined for all $t \geq 0$, the Q-transform of $f(t)$ is the function $T(u,s)$ defined by

$$T(u,s) = Q[f] = \int_0^\infty f(ut)\, e^{-\frac{t}{s}}\, dt \qquad (1)$$

Provided the integral exists for some s, where $s \in (-t_1, t_2)$.

The original function $f(t)$ in (1) is called the inverse transform or inverse of $T(u,s)$, and is denoted by

$$f(t) = Q^{-1}\{T(u,s)\}.$$

If we substitute $ut = y$, then equation (1) becomes,

$$Q[f(t)] = T(u,s) = \frac{1}{u}\int_0^\infty f(y)\, e^{-\frac{1}{us}y}\, dy \qquad (2)$$

2.1. Laplace-Q duality property:

If the Laplace transform of the function $f(t)$ is $F(s)$, then

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \qquad (3)$$

Substitute $t = uy$ in the integral on right hand side we get

$$F(s) = \mathcal{L}\{f(t)\} = u\int_0^\infty f(uy)\, e^{-suy}\, dy$$

hence, from equation (2) we get

$$F(s) = u\, T\left(u, \frac{1}{us}\right) \qquad (4)$$

also, from equations (1) and (3) we get

$$T(u,s) = \frac{1}{u}\, F\left(\frac{1}{us}\right) \qquad (5)$$

the equations (4) and (5) form the duality relation between these two transforms and may serve as a mean to get one from the other when needed.

2.1: Q- Transform of elementary functions:

The following theorems hold:

2.1.1. Polynomial function:

Theorem 1:

If $f(t) = t^n$, then its Q- Transform is

$$Q[t^n] = n!\, u^n s^{n+1}, s > 0, n \in \mathbb{N} \cup \{0\}$$

Proof:

From the definition of the Q- Transform, we have

$$Q[1] = \int_0^\infty e^{-\frac{t}{s}}\, dt = \lim_{b\to\infty}\int_0^b e^{-\frac{t}{s}} dt = s$$

$$Q[t] = \int_0^\infty ut\, e^{-\frac{t}{s}}\, dt = us^2$$

$$Q[t^2] = \int_0^\infty u^2 t^2\, e^{-\frac{t}{s}}\, dt = 2u^2 s^3$$

$$Q[t^n] = \int_0^\infty u^n t^n e^{-\frac{t}{s}}\, dt = n!\, u^n s^{n+1}$$

we generalization of above theorem in the next theorem

Theorem 2:

.If $f(t) = t^r, r > -1, s > 0$, then its Q- Transform is

$$Q[t^r] = \Gamma(r+1)\, u^r s^{r+1},$$

Where Γ is the Gamma function.

Proof:

By definition of the Q- Transform, and substitute $t = sv$, we get

$$Q[t^r] = \int_0^\infty u^r t^r e^{-\frac{t}{s}}\, dt = u^r s^{r+1}\int_0^\infty v^r e^{-v}\, dv = \Gamma(r+1)\, u^r s^{r+1}$$

2.2.2. Exponential function:

Theorem 3:

Let $f(t) = e^{at}, when\ t \geq 0$, where a is a constant, then its Q- Transform is where $\frac{1}{s} > au$.

$$Q[e^{at}] = \frac{s}{1 - a\,u\,s}$$

Proof:

$$Q[e^{at}] = \int_0^\infty e^{aut}\, e^{-\frac{t}{s}}\, dt = \lim_{b\to\infty}\int_0^b e^{-\left(\frac{1}{s} - au\right)t}\, dt = \frac{s}{1 - a\,u\,s}$$

Corollary 4:

The Q-Transform of exponential function can be written as

$$Q[e^{at}] = \begin{cases} \dfrac{1}{1-au} & \textit{Sumudu transform} \quad (s=1) \\ \dfrac{s}{1-as} & \textit{Yang transform} \quad (u=1) \end{cases}$$

Theorem 5:

The Q-Transform of the trigonometric function such as $\sin t$ and $\cos t$ yields

$$Q\{\sin at\} = \frac{au\, s^2}{1+\, a^2u^2s^2}$$

and

$$Q\{\cos at\} = \frac{s}{1+\, a^2u^2s^2}$$

where $\frac{1}{s} > au$.

2.2.3. Unit step function:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$$

$$Q[u(t)] = \int_0^\infty e^{-\frac{t}{s}}\, dt = \lim_{b\to\infty} \int_0^b e^{-\frac{t}{s}}\, dt \;= s$$

Sufficient condition for the existence of Q-Transform:

In this section we introduced the sufficient condition for the existence of Q – transform is defined over the set of functions

$$\mathcal{B} = \left\{ f(t) : \exists\, N, k_1, k_2 > 0\,, |f(t)| < N\, e^{\left(\frac{|t|}{k_j}\right)}, if\ t \in (-1)^j \times [0,\infty) \right\}$$

Theorem 6:

If $f(t)$ is sectional continuous in every finite interval $0 \le t \le b$ and of exponential order δ for $b < t$, then its Q-Transform $T(u,s)$ exists for all $s > \delta, u > \delta$.

Proof:

We have for any positive number b,

$$\int_0^\infty f(ut)\, e^{-\frac{t}{s}}\, dt = \int_0^b f(ut)\, e^{-\frac{t}{s}}\, dt + \int_b^\infty f(ut)\, e^{-\frac{t}{s}}\, dt$$

Since $f(t)$ is sectional continuous in every finite interval $0 \le t \le b$, the first integral on the right side exists. Also the second integral on the right side exists, since $f(t)$ is exponential order δ for $t > b$. To see this we have only to observe that in such case

$$\left| \int_b^\infty f(ut)\, e^{-\frac{t}{s}}\, dt \right| \le \int_b^\infty \left| f(ut)\, e^{-\frac{t}{s}} \right| dt$$

$$\le \int_0^\infty e^{-\frac{t}{s}}\, |f(u\,t)|\, dt \le M \int_0^\infty e^{-\frac{t}{s}} . e^{\delta\, u\, t}\, dt$$

$$= \frac{sM}{1 - s\,\delta\, u}$$

3. Properties of Q-Transform:

3.1. Linearity Property

Theorem 7:

Let $f(t)$ and $g(t)$ are functions, then for any constants a and b we have

$$Q\{a\, f(t) + b\, g\,(t)\} = aQ\{f(t)\} + bQ\{g(t)\}$$

Proof:

$$Q[f] = \int_0^\infty f(ut)\, e^{-\frac{t}{s}}\, dt$$

and

$$Q[g] = \int_0^\infty g(ut)\, e^{-\frac{t}{s}}\, dt$$

then for any constants a and b we get

$$Q\{a\, f(t) + b\, g(t)\} = \int_0^\infty e^{-\frac{t}{s}} \{af(ut) + b\, g(ut)\}dt = aQ\{f(t)\} + bQ\{g(t)\}$$

3.2. Change of scale property:

Theorem 8:

If $Q\{f(t)\} = T(s,u)$, then $Q\{f(at)\} = \frac{1}{a} T\left(\frac{s}{a}, u\right)$.

Proof:

$$Q\{f(at)\} = \int_0^\infty f(aut)\, e^{-\frac{t}{s}}\, dt = \frac{1}{a}\int_0^\infty f(uv)\, e^{-\frac{v}{as}}\, dv = \frac{1}{a} T\left(\frac{s}{a}, u\right).$$

3.3. ***First translation or shifting property:***

Theorem 9:

Suppose that $f(t)$ is a continuous functions and $t \geq 0$, then

$$Q\{e^{at} f(t)\} = \frac{1}{1-aus}\, T\left(\frac{u}{1-aus}, s\right).$$

Proof:

By definition of Q-Transform

$$Q\{e^{at} f(t)\} = \int_0^\infty e^{aut} f(ut)\, e^{-\frac{t}{s}}\, dt$$

$$= \frac{1}{1-aus}\int_0^\infty f\left(\frac{uw}{1-aus}\right) e^{-\frac{w}{s}}\, dw$$

$$= \frac{1}{1-aus}\, T\left(\frac{u}{1-aus}, s\right)$$

3.4. ***Q-Transform of derivatives:***

Theorem 10:

Let $Q\{f(t)\} = T(u,s)$, then

$$Q\{f^{(n)}(t)\} = \frac{T(u,s)}{u^n s^n} - \frac{1}{u}\sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{(us)^{n-k-1}}$$

Proof:

Integrated by parts we get

$$Q\{f'(t)\} = \int_0^\infty e^{-\frac{t}{s}} \frac{d}{dt} f(ut)\, dt = \frac{1}{us} T(u,s) - \frac{1}{u} f(0)$$

By mathematical induction we get the result.

The proof is complete. □

3.5. *Q-Transform of integrals:*

Theorem 11:

If $f(t)$ is sectional continuous in every finite interval $0 \leq t \leq b$ and of exponential order δ for $b < t$, Let $Q\{f(t)\} = T(u,s)$, then

$$Q\left\{\int_0^t f(v)\, dv\right\} = u\, s\, T(u,s)$$

***Proof*:**

Let

$$G(t) = \int_0^t f(v)\, dv$$

then

$$G^{'}(t) = f(t), \quad G(0) = 0$$

taking the Q-Transform of both sides, and by theorem (10) with $n = 1$, we get

$$Q(G'(t)) = Q\{f(t)\}$$

$$\frac{1}{us} Q(G(t)) - \frac{G(0)}{u} = T(u,s)$$

This proof is complete. □

4. Application of Q- Transform:

The covering equation for temperature distributions in a semi-infinite bar is

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad ; \qquad (6)$$

with the boundary and initial conditions

$$v(0,t) = v(1,t) = 0$$

$$v(x,0) = 3\sin 2\pi x$$

we can think of this initial – boundary value problem as modeling the temperature distribution in a homogeneous bar of material lying along the positive x – axis, with a control function on the temperature.

Apply the Q-transform with respect to time to the heat equation to obtain

$$\frac{1}{us} V(x,s,u) - \frac{1}{u} v(x,0) = \frac{d^2V(x,s,u)}{dx^2}$$

$$\frac{d^2V(x,s,u)}{dx^2} - \frac{1}{us} V(x,s,u) = -\frac{3}{u}\sin 2\pi x$$

the general solution of equation (6) can be written as

$$V(x,s,u) = V_c(x,s,u) + V_p(x,s,u)$$

the characteristic equation of the above differential equation is

$$r = \pm \frac{1}{\sqrt{us}}$$

since the roots are real and distinct, therefore, the solution is of the form

$$V_c(x,s,u) = c_1 e^{\frac{1}{\sqrt{us}}x} + c_2 e^{-\frac{1}{\sqrt{us}}x}. \qquad (7)$$

Now, we will find c_1 and c_2 by using boundary conditions. Equation (7) becomes

$$V_c(x,s,u) = 0.$$

Now, we find the solution of nonhomogeneous part $V_p(x,s,u)$ which is given by

$$V_p(x,s,u) = c_3 \sin 2\pi\, x + c_4 \cos 2\pi\, x$$

using the method of undetermined coefficients on the nonhomogeneous part, we get

$$V_p(x,s,u) = \frac{3\,s}{1+4\,\pi^2 us} \sin 2\pi\, x.$$

Now, we obtain the general solution of equation (2) is

$$V(x,s,u) = \frac{3\,s}{1+4\,\pi^2 us} \sin 2\pi\, x \qquad (8)$$

taking the inverse Q - transform of Eq. (8), we get

$$v(x,t) = 3\, e^{-4\pi^2 t} \sin 2\,\pi\, x$$

5. Conclusion:

A new transform is introduced and its properties are mentioned. It is demonstrated that the theorems related to Laplace and Sumudu are also true for this new transform. Finally, the new transformed is applied to temperature distributions in a semi-infinite bar problem.

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